The Forward Kinematics of 3-RPR Planar Robots: A Review and a Distance-Based Formulation

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***Abstract*—The standard forward kinematics analysis of 3-RPR planar parallel robots boils down to computing the roots of a sextic polynomial. There are many different ways to obtain this polynomial but most of them include exceptions for which the formulation is not valid. Unfortunately, near these exceptions the corresponding polynomial exhibits numerical instabilities. In this paper, we provide a way around this inconvenience by translating the forward kinematics problem to be solved into an equivalent problem fully stated in terms of distances. Using constructive geometric arguments, an alternative sextic —which is not linked to a particular reference frame— is straightforwardly obtained without the need of variable eliminations nor tangent-half-angle substitutions. The presented formulation is valid, without any modification, for any planar 3-RPR parallel robot, including the special architectures and configurations —which ultimately lead to numerical instabilities— that cannot be directly handled by previous formulations.**

***Index Terms*—3-RPR parallel robots, position analysis, forward kinematics, coordinate-free formulations, Cayley-Menger determinants, bilateration**

I. Introduction

Much has been written about the 3-RPR planar parallel robot because of its practical interest, mechanical simplicity, and rich mathematical properties [1]. Such a robot consist of a moving platform connected to the ground through three revolute-prismatic-revolute kinematic chains. The prismatic joint of each chain is actuated and the forward kinematics problem consists in, given the prismatic joint lengths, calculating the Cartesian pose of the moving platform. A clever reasoning, based on the number of possible intersections between a circle and the general coupler curve of a 4-bar mechanism, permits to conclude that this problem has at most 6 different solutions [2]. That is, for fixed leg lengths, it is possible to assemble the robot in up to six different ways, known as *assembly modes*. In general, it is not possible to express analytically these six Cartesian poses as functions of the actuated joint coordinates, except for some particular cases known as *analytic robots* [3]. This paper is devoted to the problem of finding these poses efficiently and accurately for all cases.

The usual approach to obtain the aforementioned assembly modes consists in manipulating the kinematic equations of the robot to reduce the problem to finding the roots of a polynomial in one variable, the *characteristic polynomial*, which must be of the lowest possible degree, that is, a sextic. E. Peysah is credited to be the first researcher in obtaining this sextic in 1985 [4]. The same result was obtained independently at least by G. Pennock and D. Kassner in 1990 [5], K. Wohlhart in 1992 [6], and C. Gosselin *et al.*, also in 1992 [7]. The formulation due to C. Gosselin *et al.* has become thereafter the standard one. The major step in this formulation is to find an equation only in θ (the orientation of the moving platform), that is, to eliminate all other variables from the system until an equation is obtained that contains only θ. Finally, a tangent-half-angle substitution is applied to translate sine and cosine functions of θ into rational polynomial expressions in a new variable $t = \tan(\theta/2)$.

In order to simplify as much as possible the coefficients of the resulting 6th-degree polynomial, it is possible to express the coordinates of the base attachments according to a specific coordinate frame. For example, by making one coordinate axis to coincide with the baseline between two base attachments and/or locating the origin at one base attachment. Nevertheless, this kind of simplifications has an important drawback: the numerical conditioning of the resulting formulation depends on the chosen reference frame. This is why those formulations which are not linked to a particular reference frame — or coordinate-free formulations— are preferable. In 2001, X. Kong and C. Gosselin proposed a coordinate-free formulation by deriving a sextic in $\tan(\psi/2)$, where ψ is the angle formed between one leg and one of its adjacent base sides [8]. Although this formulation was used to study analytic instances, it is certainly superior to the one in [7] for the aforementioned reason. Nevertheless, the problems derived from the tangent-half-angle substitution still remained.

The tangent-half-angle substitution poses two well-known problems. One results from the fact that $\tan(\theta/2)$ is undefined for $\theta = \pm\pi$. Moreover, it can become difficult to reconstruct other roots, occurring in conjunction with the root $\theta = \pm\pi$ [9]. The other problem is the introduction of extraneous roots. Both problems are well known and can be handled but it complicates notably subsequent calculations [10]. One alternative to this substitution is to keep $\cos(\theta)$ and $\sin(\theta)$, both as variables, and to add the equation $\sin^2(\theta) + \cos^2(\theta) = 1$ to the elimination process.

A more elegant mathematical framework is obtained by viewing the planar moving platform displacements as points in a four-dimensional homogeneous space. This can be achieved using, for example, the kinematic mapping, as in [11] and further elaborated in [12], or Clifford algebra, as in [13]. A similar treatment may be obtained by using the substitutions $\sin(\theta) = 2sc/(c^2 + s^2)$ and $\cos(\theta) = (c^2 - s^2)/(c^2 + s^2)$ which, after clearing denominators, lead to homogeneous equations in s and c. This non coordinate-free formulations avoid the tangent-half-angle substitution but the problem with $\pm\pi$ turns still remains if one of the used homogeneous coordinates is normalized to 1. Alternatively, a normalizing condition involving two variables is possible thus adding one more equation to the elimination process.

An important fact that has been commonly overlooked by the kinematics community is that solving the forward kinematics of the 3-RPR parallel robot is equivalent to finding the distinct planar embeddings, up to Euclidean motions, of a graph with vertices subject to edge lengths constraints. This graph corresponds to what in [14] is called the *doublet*, or in [15], the *Desargues framework*. In both cases, the number of possible embeddings is obtained by formulating the problem purely in terms of distances. This kind of approach leads to undesired solutions to the original problem because the embeddings containing mirror reflections of the base and/or the moving platform also count as valid solutions. In [14], the embedding problem is tackled by assigning coordinates to two points whose distance is known and solving a system of 8 equations (the remaining 8 distances constraints) in 8 variables (the coordinates of the remaining 4 points). The resultant is a polynomial of degree 28 which factors as the product of a degree 12 and a degree 16 polynomial. Alternatively, in [15], the problem is formulated in terms of equations involving Cayley-Menger determinants which permit to conclude that there exists edge lengths which induce up to 24 embeddings, 6 for each combination of the base and the platform triangles and their mirror reflections. In this paper, we introduce a further twist to this approach that allows us to solve the problem by a sequence of bilaterations following the initial ideas presented in

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This paper has supplementary downloadable multimedia material. This material includes several Maple Worksheets. For each of the examples in Section V a worksheet is included detailing a particular numerical example. No particular requirements, except for an installed copy of Maple, version 12.0 or higher. Contact the first author for further questions about this material.

[16]. As a result, a 6th-degree characteristic polynomial, which is not linked to any particular reference frame, is straightforwardly obtained without variable eliminations nor tangent-half-angle substitutions. Moreover, the obtained polynomial is mathematically more tractable than the one obtained using other approaches because its coefficients are the result of operating with Cayley-Menger determinants with geometric meaning.

This paper is organized as follows. A coordinate-free formula for bilateration expressed in terms of Cayley-Menger determinants is presented in Section II. It is the basic formula, used in Section III, to derive a distance-based characteristic polynomial for the general 3-R$\underline{\text{P}}$R planar parallel robot. Section IV discusses how this formulation specializes to all analytic instances reported in the literature. Section V analyzes several numerical examples. Finally, Section VI summarizes the main points.

II. Cayley-Menger determinants and bilateration

Let P_i and $\mathbf{p}_i$ denote a point and its position vector in a given reference frame, respectively. Then, let us define

$$D(i_1,\ldots,i_n;j_1,\ldots,j_n) = 2\left(-\frac{1}{2}\right)^n \begin{vmatrix} 0 & 1 & \ldots & 1 \\ 1 & s_{i_1,j_1} & \ldots & s_{i_1,j_n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & s_{i_n,j_1} & \ldots & s_{i_n,j_n} \end{vmatrix}, \tag{1}$$

with $s_{i,j} = \|\mathbf{p}_i - \mathbf{p}_j\|^2$, which is independent from the chosen reference frame. This determinant is known as the *Cayley-Menger bi-determinant* of the point sequences $P_{i_1},\ldots,P_{i_n}$, and $P_{j_1},\ldots,P_{j_n}$. When the two point sequences are the same, it will be convenient to abbreviate $D(i_1,\ldots,i_n;i_1,\ldots,i_n)$ by $D(i_1,\ldots,i_n)$, which is simply called the *Cayley-Menger determinant* of the involved points.

In terms of Cayley-Menger determinants, the squared distance between P_i and P_j can be expressed as $D(i,j)$ and the signed area[1] of the triangle $P_iP_jP_k$, as $\pm\frac{1}{2}\sqrt{D(i,j,k)}$. For a brief review of the properties of Cayley-Menger determinants, see [17].

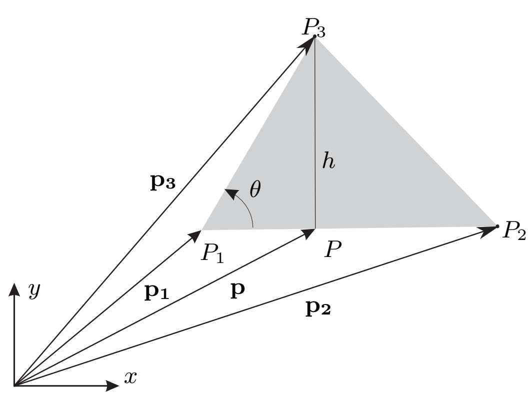


Fig. 1. The bilateration problem in $\mathbb{R}^2$.

The bilateration problem in $\mathbb{R}^2$ consists of finding the feasible locations of a point, say P_3, given its distances to two other points, say P_1 and P_2, whose locations are known. Then, according to Fig. 1, the position vector of the orthogonal projection of P_3 onto the line P_1P_2 can be expressed as:

$$\begin{aligned}\mathbf{p} &= \mathbf{p}_1 + \sqrt{\frac{D(1,3)}{D(1,2)}}\cos\theta(\mathbf{p}_2 - \mathbf{p}_1) \\ &= \mathbf{p}_1 + \frac{D(1,2;1,3)}{D(1,2)}(\mathbf{p}_2 - \mathbf{p}_1).\end{aligned} \tag{2}$$

Moreover, the position vector of P_3 can be expressed as:

$$\mathbf{p}_3 = \mathbf{p} \pm \frac{\sqrt{D(1,2,3)}}{D(1,2)}\mathbf{S}(\mathbf{p}_2 - \mathbf{p}_1), \tag{3}$$

where the $\pm$ sign accounts for the two mirror symmetric locations of P_3 with respect to the line defined by P_1P_2, and $\mathbf{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then, substituting (2) in (3) and expressing the result in matrix form, we obtain

$$(\mathbf{p}_3 - \mathbf{p}_1) = \mathbf{Z}_1(\mathbf{p}_2 - \mathbf{p}_1) \tag{4}$$

$$(\mathbf{p}_3 - \mathbf{p}_2) = \mathbf{Z}_2(\mathbf{p}_1 - \mathbf{p}_2) \tag{5}$$

where

$$\mathbf{Z}_1 = \frac{1}{D(1,2)}\begin{bmatrix} D(1,2;1,3) & \mp\sqrt{D(1,2,3)} \\ \pm\sqrt{D(1,2,3)} & D(1,2;1,3) \end{bmatrix},$$

and

$$\mathbf{Z}_2 = \frac{1}{D(2,1)}\begin{bmatrix} D(2,1;2,3) & \mp\sqrt{D(2,1,3)} \\ \pm\sqrt{D(2,1,3)} & D(2,1;2,3) \end{bmatrix}.$$

Substituting (4) and (5) in the vector loop equation

$$(\mathbf{p}_3 - \mathbf{p}_1) + (\mathbf{p}_2 - \mathbf{p}_3) + (\mathbf{p}_1 - \mathbf{p}_2) = 0, \tag{6}$$

it is possible to conclude that $\mathbf{Z}_1 + \mathbf{Z}_2 = \mathbf{I}$. $\mathbf{Z}_1$ and $\mathbf{Z}_2$ will be called bilateration matrices. Since they are of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, their product commutes. Actually, this kind of matrices constitute an Abelian group under product and addition. Moreover, if $\mathbf{v} = \mathbf{Z}\mathbf{w}$, where $\mathbf{Z}$ is a bilateration matrix, then it can be checked that $||\mathbf{v}||^2 = \det(\mathbf{Z})\,||\mathbf{w}||^2$.

III. Distance-Based Formulation

Fig. 2 shows a general 3-R$\underline{\text{P}}$R planar parallel robot. The center of the three grounded passive revolute joints define the base triangle $P_1P_2P_3$ and the three moving passive revolute joints centers, the moving triangle $P_4P_5P_6$. The active prismatic joint variables are the lengths ρ_1, ρ_2, and ρ_3. Angles α and β have been chosen so that their signs determine the orientation of the base and platform triangles.

Next, we derive a coordinate-free formula for the forward kinematics of this parallel robot. To this end, instead of directly computing the Cartesian pose of the moving platform, first we will compute the set of values of $T = ||\mathbf{p}_1 - \mathbf{p}_5||^2$ compatible with ρ_1, ρ_2, and ρ_3 and the base and the moving platform side lengths, L_1, L_2, L_3, and l_1, l_2, l_3, respectively. Thus, this step is entirely posed in terms of distances.

According to Fig. 3, we have

$$(\mathbf{p}_6 - \mathbf{p}_5) = \mathbf{B}(\mathbf{p}_4 - \mathbf{p}_5) = \mathbf{B}\mathbf{A}(\mathbf{p}_1 - \mathbf{p}_5), \tag{7}$$

and

$$(\mathbf{p}_3 - \mathbf{p}_1) = \mathbf{D}(\mathbf{p}_2 - \mathbf{p}_1) = \mathbf{D}\mathbf{C}(\mathbf{p}_5 - \mathbf{p}_1), \tag{8}$$

where

$$\begin{aligned}\mathbf{B} &\triangleq \begin{bmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{bmatrix} \\ &= \frac{1}{D(1,2)}\begin{bmatrix} D(1,2;1,3) & -\text{sign}(\alpha)\sqrt{D(1,2,3)} \\ \text{sign}(\alpha)\sqrt{D(1,2,3)} & D(1,2;1,3) \end{bmatrix}\end{aligned}$$

[1] For a triangle $P_iP_jP_k$ in the Euclidean plane with area A, the *signed area* is defined as $+A$ (respectively, $-A$) if the point P_j is to the right (respectively to the left) of the line P_iP_k, when going from P_i to P_k

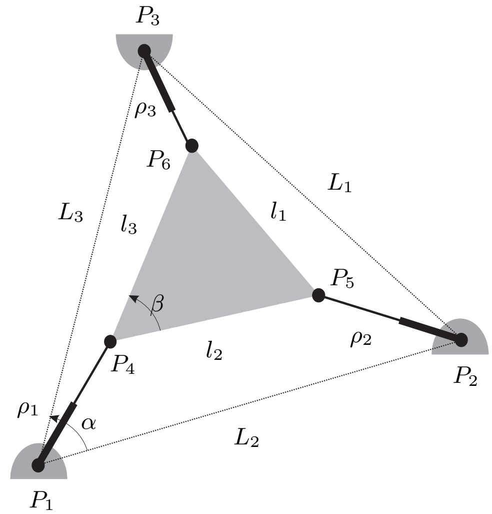


Fig. 2. A general planar 3-R<u>P</u>R parallel robot and its associated notation.

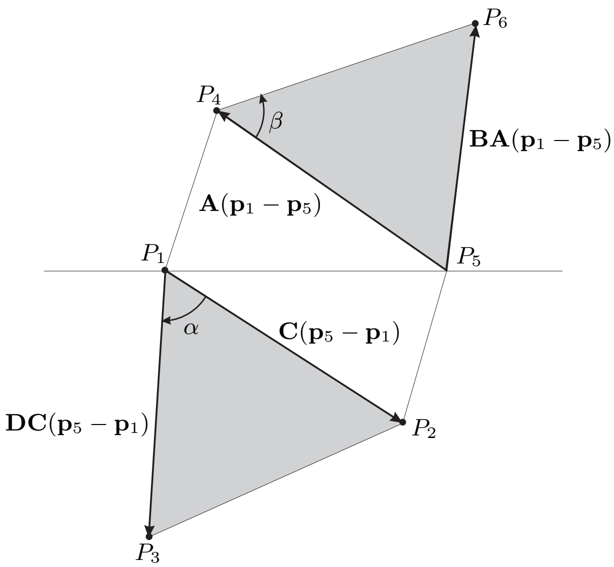


Fig. 3. $(\mathbf{p}_6-\mathbf{p}_3)$ can be expressed in function of $(\mathbf{p}_5-\mathbf{p}_1)$ by computing four bilaterations.

and

$$\mathbf{D} \triangleq \begin{bmatrix} d_1 & -d_2 \\ d_2 & d_1 \end{bmatrix} = \frac{1}{D(5,4)} \begin{bmatrix} D(5,4;5,6) & \operatorname{sign}(\beta)\sqrt{D(5,4,6)} \\ -\operatorname{sign}(\beta)\sqrt{D(5,4,6)} & D(5,4;5,6) \end{bmatrix}$$

are constant matrices that depend only on the geometry of the base and the moving platform, respectively, and

$$\mathbf{A} = \frac{1}{D(1,5)} \begin{bmatrix} D(1,5;1,2) & \mp\sqrt{D(1,5,2)} \\ \pm\sqrt{D(1,5,2)} & D(1,5;1,2) \end{bmatrix}$$

and

$$\mathbf{C} = \frac{1}{D(5,1)} \begin{bmatrix} D(5,1;5,4) & \mp\sqrt{D(5,1,4)} \\ \pm\sqrt{D(5,1,4)} & D(5,1;5,4) \end{bmatrix}$$

that are function of $T = D(1,5) = D(5,1)$. Now, by substituting (7) and (8) in the vector loop equation

$$(\mathbf{p}_5-\mathbf{p}_1)+(\mathbf{p}_6-\mathbf{p}_5)+(\mathbf{p}_3-\mathbf{p}_6)+(\mathbf{p}_1-\mathbf{p}_3)=0, \quad (9)$$

we obtain

$$(\mathbf{p}_3-\mathbf{p}_6)=\mathbf{\Omega}(\mathbf{p_5}-\mathbf{p_1}) \quad (10)$$

where $\mathbf{\Omega}=\mathbf{I}-(\mathbf{AB}+\mathbf{CD})$. This matrix, when expanded in terms of Cayley-Menger determinants, leads to:

$$\mathbf{\Omega} = \frac{1}{D(1,5)} \begin{bmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{bmatrix} \quad (11)$$

where

$$\begin{aligned} w_1 =& D(1,5)-D(1,5;1,2)b_1-D(5,1;5,4)d_1 \\ & \mp\sqrt{D(1,5,2)}b_2 \mp \sqrt{D(5,1,4)}d_2 \end{aligned} \quad (12)$$

$$\begin{aligned} w_2 =& -D(1,5;1,2)b_2-D(5,1;5,4)d_2 \\ & \pm\sqrt{D(1,5,2)}b_1 \pm \sqrt{D(5,1,4)}d_1. \end{aligned} \quad (13)$$

Since

$$\det(\mathbf{\Omega}) = \frac{||\mathbf{p_3}-\mathbf{p_6}||^2}{||\mathbf{p_5}-\mathbf{p_1}||^2},$$

then

$$w_1^2+w_2^2=D(3,6)D(1,5). \quad (14)$$

Now, if (12) and (13) are substituted in (14) and all the involved Cayley-Menger determinants are expanded in terms of distances, we obtain

$$\Phi_a+\Phi_b A_1+\Phi_c A_2+\Phi_d A_1 A_2=0 \quad (15)$$

where

$$A_1 = \pm\frac{1}{2}\sqrt{[T-(L_2-\rho_2)^2]\,[(L_2+\rho_2)^2-T]}$$
$$A_2 = \pm\frac{1}{2}\sqrt{[T-(l_2-\rho_1)^2]\,[(l_2+\rho_1)^2-T]}$$

and

$$\begin{aligned} \Phi_a =& \left(\frac{1}{2}b_1 d_1+\frac{1}{2}b_2 d_2-b_1-d_1+1\right)T^2 \\ &+ \Big[L_2^2(b_1^2+b_2^2)+l_2^2(d_1^2+d_2^2) \\ &+ \frac{1}{2}\left(-\rho_1^2+l_2^2+L_2^2-\rho_2^2\right)(b_1 d_1+b_2 d_2) \\ &+ \left(\rho_2^2-L_2^2\right)b_1+\left(\rho_1^2-l_2^2\right)d_1-\rho_3^2\Big]\,T \\ &+ \frac{1}{2}(l_2^2-\rho_1^2)(L_2^2-\rho_2^2)(b_1 d_1+b_2 d_2) \\ \Phi_b =& (b_1 d_2-b_2 d_1+2b_2)T+(l_2^2-\rho_1^2)(b_1 d_2-b_2 d_1) \\ \Phi_c =& (b_2 d_1-b_1 d_2+2d_2)T+(L_2^2-\rho_2^2)(b_2 d_1-b_1 d_2) \\ \Phi_d =& 2(b_1 d_1+b_2 d_2) \end{aligned}$$

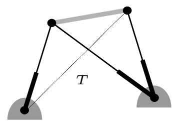
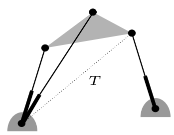
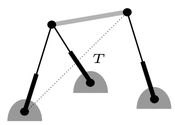
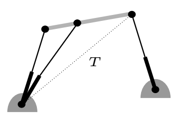
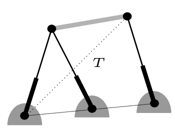
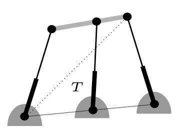
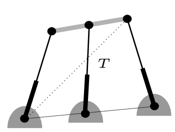
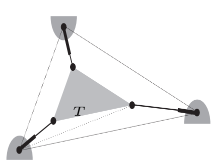
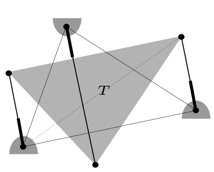
Equation (15) is a scalar radical equation in T whose roots, that are in the range for which the two square roots in the definition of A_1 and A_2 yield real values, i.e., the range

$$\left[\max\{(L_2-\rho_2)^2,(l_2-\rho_1)^2\},\min\{(L_2+\rho_2)^2,(l_2+\rho_1)^2\}\right], \quad (16)$$

determine the assembly modes of the analyzed robot. These roots can be readily obtained for the four possible combinations of signs for A_1 and A_2 using, for example, a Newton interval method. In order to obtain a polynomial representation, the squared roots in (15) can be eliminated by properly twice squaring it. This operation yields

$$\begin{aligned} &-\Phi_d^4 A_1^4 A_2^4+2\Phi_d^2\Phi_b^2 A_1^4 A_2^2+2\Phi_d^2\Phi_c^2 A_1^2 A_2^4 \\ &-\Phi_b^4 A_1^4-\Phi_c^4 A_2^4+2\Phi_a^2\Phi_b^2 A_1^2+2\Phi_a^2\Phi_c^2 A_2^2 \\ &+\left(-8\Phi_b\Phi_c\Phi_d\Phi_a+2\Phi_b^2\Phi_c^2+2\Phi_d^2\Phi_a^2\right)A_1^2 A_2^2-\Phi_a^4=0 \end{aligned}$$

TABLE I
THE KNOWN 3-R$\underline{P}$R ANALYTIC PLANAR ROBOTS

Case		**Φ-polynomials**	**Degree of $\Gamma(T)$**	**Previous works**
Double Coincidence		$\Phi_a = T^2 + b\,T$ $\Phi_b = 0$ $\Phi_c = 0$ $\Phi_d = 0$	1	
Coincidence		$\Phi_a = a\,T^2 + b\,T$ $\Phi_b = 0$ $\Phi_c = f\,T$ $\Phi_d = 0$	2	Collins [13] (1 quartic)
		$\Phi_a = a\,T^2 + b\,T$ $\Phi_b = d\,T$ $\Phi_c = 0$ $\Phi_d = 0$	2	
Coincidence + Collinearity		$\Phi_a = a\,T^2 + b\,T$ $\Phi_b = 0$ $\Phi_c = 0$ $\Phi_d = 0$	1	Collins [13] (1 quadratic) Gosselin & Merlet [3] (2 quadratics)
		$\Phi_a = a\,T^2 + b\,T$ $\Phi_b = 0$ $\Phi_c = 0$ $\Phi_d = 0$	1	
Collinearity		$\Phi_a = a\,T^2 + b\,T + c$ $\Phi_b = 0$ $\Phi_c = 0$ $\Phi_d = h$	3	Collins [13] (1 cubic) Gosselin & Merlet [3] (1 sextic) Kong & Gosselin [8] (1 cubic + 1 quadratic)
Collinearity + Similarity		$\Phi_a = a\,T^2 + b\,T + c$ $\Phi_b = 0$ $\Phi_c = 0$ $\Phi_d = 4\,a$	2	Kong & Gosselin [8] (2 quadratics)
Similarity		$\Phi_a = a\,T^2 + b\,T + c$ $\Phi_b = d\,T + e$ $\Phi_c = -d\,T + g$ $\Phi_d = 4\,a$	4	Kong & Gosselin [8] (2 quadratics) Gosselin & Merlet [3] (1 cubic + 1 quadratic) Collins [13] (1 quadratic) Ji & Wu [18] (2 quadratics)
Mirror reflection		$\Phi_a = a\,T^2 + b\,T + c$ $\Phi_b = d\,T + e$ $\Phi_c = f\,T + g$ $\Phi_d = 4\,a$	6 (solvable)	Wenger *et al.* [20] (1 cubic + 1 quadratic)

which, when fully expanded, leads to an expression of the form:

$$T^2\Gamma(T) = 0 \tag{17}$$

where $\Gamma(T)$ is a 6th-degree polynomial[2] in T. The double extraneous root at $T = 0$ was introduced when clearing denominators to obtain (14), so it can be dropped.

IV. Analytic Robots

The leading coefficients of Φ_a, Φ_b, Φ_c, and Φ_d do not depend on ρ_1, ρ_2, or ρ_3. As a consequence, they can be made to be identically zero by properly choosing the dimensions of the base and the moving platform thus simplifying the formulation. For example, the maximum simplification is attained by coalescing two attachments both in the base and the platform. In this case, $\Phi_a = T^2 + bT$ and $\Phi_b = \Phi_c = \Phi_d = 0$. Table I compiles different geometric conditions that lead to simplifications for the resulting characteristic polynomial. All of them have already been studied on a case-by-case basis [3], [8], [13], [18], [20]. They lead to analytic robots because the roots of the resulting characteristic polynomials can be obtained using only the basic arithmetic operations and the taking of n-th roots. Table I summarizes, for each case, the resulting Φ-polynomials, the degree of the characteristic polynomial derived in the previous section, and references to related works. Since, in general, these related works use ad-hoc formulations that require solving more than one polynomial in cascade, the degrees of these polynomials are given in parenthesis besides the corresponding reference.

There have been found four families of analytic robots that satisfy at least one of the following geometric conditions:

C1: two attachments on the base, or on the platform, coincide;
C2: the attachments, both on the base and the platform, are collinear;
C3: the base and platform triangles are similar; and
C4: the base and the platform are inverted triangles (one is the mirror reflection of the other).

It is well-known that there are formulas involving radicals for finding the roots of polynomials of degree lower than 5. As a consequence, the analytic 3-R$\underline{\text{P}}$R planar parallel robots are also referred as those robots whose characteristic polynomial is of degree lower than 5 or it factors into terms of degree lower than 5. Nevertheless, it can be checked that the irreducible characteristic polynomial in T for a parallel robot satisfying the geometric condition C4 —which is known to be analytic— is of degree 6. The solution to this apparent contradiction requires Galois theory. To be precise, we recall that a polynomial equation is solvable by radicals precisely when the Galois group of the polynomial is solvable. It can be checked that the resulting sextic in T for a parallel platform satisfying the geometric condition C4 is solvable [21]. Thus, a more precise definition of analytic robots would be “robots whose characteristic polynomial Galois group is solvable.”

V. Examples

The examples contained in this section try to highlight the advantages of the proposed distance-based formulation, first by analyzing a case in which the standard previous formulations fail to provide the correct result, and then by showing that it is valid for all specialized cases that have been previously studied on a ad hoc basis. The numerical details can be found it the attached supplementary multimedia material

A. Example I: A comparison with previous formulations

Let us study the planar 3-R$\underline{\text{P}}$R parallel robot with geometric parameters $\alpha > 0$, $\beta > 0$, $l_1 = 5$, $l_2 = 6$, $l_3 = 5$, $L_1 = \sqrt{73}$,

[2]The expression for this polynomial can be found in the attached multimedia material.

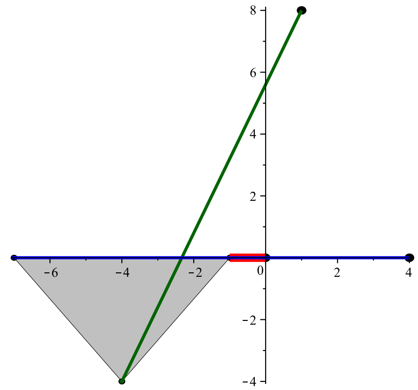


Fig. 4. Configuration analyzed in Example I using the formulations presented in [7], [8], and [11]. The lines in red, blue, and green correspond to the legs defined by P_1P_4, P_2P_5, and P_3P_6, respectively.

$L_2 = 4$, and $L_3 = \sqrt{65}$, and input joints $\rho_1 = 1$, $\rho_2 = 11$, and $\rho_3 = 13$. If $\mathbf{p}_1 = (0,0)^T$, $\mathbf{p}_2 = (4,0)^T$, and $\mathbf{p}_3 = (1,8)^T$, it can be verified that the characteristic polynomial of this robot, using the formulation derived in [7], reduces to:

$$\begin{aligned}&1469440\,X^4 + 1755136\,X^3 + 4261376\,X^2\\&+ 1140736\,X + 219136\end{aligned} \tag{18}$$

with

$$\sin(\theta) = \frac{2\,X}{1+X^2} \quad\text{and}\quad \cos(\theta) = \frac{1-X^2}{1+X^2},$$

θ being the angle between the lines defined by P_1P_2 and P_4P_5. The roots of this polynomial are $-0.4573-1.5419i$, $-0.4573+1.5419i$, $-0.1399-0.1952i$, and $-0.1399+0.1952i$. Since none of them is real, it can be erroneously concluded that the robot under study cannot be assembled with the given leg lengths.

Alternately, using the formulation derived in [8], the following characteristic polynomial is obtained:

$$\begin{aligned}&4408320Y^4 - 1744896Y^3 + 7788032Y^2\\&- 1464320Y + 3564544\end{aligned}$$

where

$$\sin(\psi) = \frac{2Y}{1+Y^2} \quad\text{and}\quad \cos(\psi) = \frac{1-Y^2}{1+Y^2},$$

ψ being the angle between the lines defined by P_1P_4 and P_1P_2. The roots of this polynomial are $-0.0363-0.9243i$, $-0.0363+0.9243i$, $0.2342-0.9435i$, and $0.2342+0.9435i$. Again, since none of them is real, it can be erroneously concluded that the robot under study cannot be assembled with the given leg lengths thus confirming the results obtained using the formulation proposed in [7]. The formulation described in [11] leads to an analogous situation when one of the homogeneous coordinates is normalized to 1. Using the implementation for this formulation reported in [22], and choosing the moving reference frame such that $\mathbf{p}_4 = (0,0)^T$ and $\mathbf{p}_5 = (6,0)^T$ in it, the resulting polynomial is:

$$\begin{aligned}&1469440Z^4 + 1755136Z^3 + 4261376Z^2\\&+ 1140736Z + 219136,\end{aligned}$$

where Z is a component of the kinematic image space coordinates (referenced as x_1 in [22]). The roots of this polynomial are $-0.4573-1.5419i$, $-0.4573+1.5419i$, $-0.1399-0.1952i$, and $-0.1399+0.1952i$. Again, none of them is real. Nevertheless, substituting the geometric parameters of the robot under study and the values of the input variables given above in the polynomial derived in Section III, the following characteristic polynomial is obtained

$$\begin{aligned}&-7738000T^6+4843775840T^5-1068953603696T^4\\&+100805055226688T^3-4600887845553776T^2\\&+101331227980892000T-876950498856250000.\end{aligned}$$

The roots of this polynomial are $27.4034-8.5802i$, $27.4034+8.5802i$, $236.5829-35.6700i$, $236.5829+35.6700i$, and a double root at 49.0000. It can be checked that the obtained double real root corresponds to a valid configuration of the analyzed 3-R$\underline{\text{P}}$R parallel robot, in clear contradiction with what was concluded using the formulations proposed in [7], [8], and [11]. In the moving platform pose associated with this double root, $\theta=\pi$, $\psi=\pi$, and $\mathbf{p}_4=(-1,0)^T$. Fig. 4 depicts this configuration.

The obtained results confirm that the previous formulations might incur into robustness problems. This is a highly relevant fact for the kinematic analysis and non-singular assembly-mode change studies of 3-R$\underline{\text{P}}$R parallel robots [23], [24]. The presented distance-based formulation does not exhibit this kind of undesirable behavior.

B. Example II: Roots at $T=0$

Consider the robot with geometric parameters $\alpha>0$, $\beta>0$, $l_1=\sqrt{13}$, $l_2=4$, $l_3=\sqrt{13}$, $L_1=5$, $L_2=4$, and $L_3=3$, and input joints $\rho_1=4$, $\rho_2=4$, and $\rho_3=2$. Substituting these values in $\Gamma(T)$, the following polynomial is obtained

$$-83200T^6+5603328T^5-84934656T^4.$$

It has a quadruple root at $T=0$ that leads to two valid configurations. The moving platform poses associated with each root of the above polynomial for the case in which $\mathbf{p}_1=(1,0)^T$, $\mathbf{p}_2=(2\sqrt{3}+1,2)^T$, and $\mathbf{p}_3=(-\frac{1}{2},\frac{3}{2}\sqrt{3})^T$ appear in Fig. 5.

Analogously to the previous example, this one cannot either be properly analyzed using the formulation presented in [8] and, depending on the location of the chosen reference frames, using the formulations derived in [7] and [11].

Finally, observe that, if $T=0$, the moving platform pose can be obtained by only two bilaterations which determine up to four possible values for $\mathbf{p}_6$ and at least one of them must satisfy the distance constraint between P_3 and P_6.

C. Example III: Coalescence of two attachments

Consider the manipulator with geometric parameters $\alpha>0$, $\beta<0$, $l_1=\sqrt{8}$, $l_2=\sqrt{10}$, $l_3=\sqrt{10}$, $L_1=L_2=5$, and $L_3=0$, and input joints $\rho_1=\sqrt{10}$, $\rho_2=5$, and $\rho_3=6$. Substituting these values in $\Gamma(T)$, the following polynomial is obtained:

$$10T^2-592T+7840,$$

whose roots are 20 and $\frac{196}{5}$. Each of them have two associated moving platform poses. The four resulting poses for the case in which $\mathbf{p}_1=\mathbf{p}_3=(5,0)^T$ and $\mathbf{p}_2=(0,0)^T$ appear in Fig. 6.

D. Example IV: Collinearity of base and platform attachments

The collinear of the base and platform attachments imply that $l_2\pm l_1\pm l_3=0$ and $L_2\pm L_1\pm L_3=0$ for a certain combination of signs. As an example, consider the robot with geometric parameters $\alpha>0$, $\beta>0$, $l_1=1$, $l_2=3$, $l_3=2$, $L_1=1$, $L_2=1$, and $L_3=2$, and input joints $\rho_1=1$, $\rho_2=2$, and $\rho_3=2$. This robot also used as an example in [7] and [8]. Substituting these values in $\Gamma(T)$, the following polynomial is obtained

$$8T^3-78T^2+195T-44,$$

T	θ	$\mathbf{p}_4$
0.0000	−0.6524	$(-2.1785, 2.4284)^T$
0.0000	0.5236	$(-2.4641, -2.0000)^T$
23.0400	−0.7634	$(4.4641, 2.0000)^T$
44.3077	−1.4420	$(4.4641, 2.0000)^T$

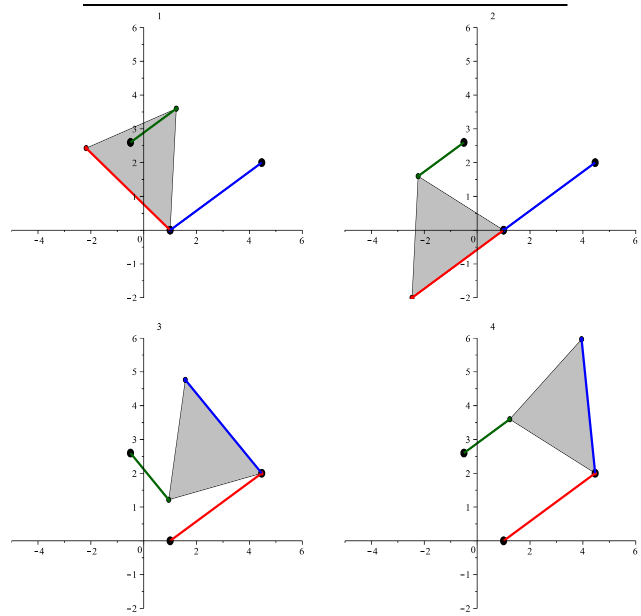


Fig. 5. The four moving platform poses obtained in Example II and their graphical representation.

T	θ	$\mathbf{p}_4$
20	−1.2490	$(2.0000, -1.0000)^T$
20	−0.3218	$(6.0000, 3.0000)^T$
$\frac{196}{5}$	1.0362	$(2.6913, -2.1610)^T$
$\frac{196}{5}$	−0.7524	$(3.3887, 2.7210)^T$

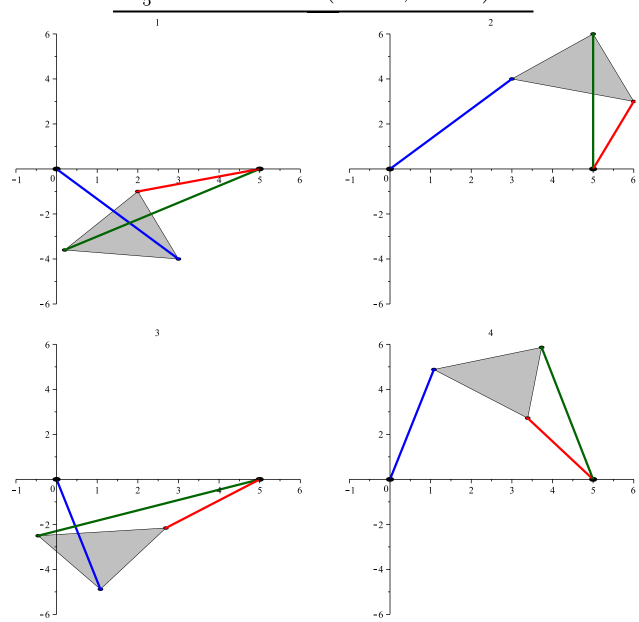


Fig. 6. The four moving platform poses obtained in Example III and their graphical representation.

T	θ	$\mathbf{p}_4$
4	1.3181	$(-0.2500, -0.9682)^T$
4	-1.3181	$(-0.2500, 0.9682)^T$
$\frac{11}{2}$	0.7227	$(-1.0000, 0.0000)^T$
$\frac{11}{2}$	-0.7227	$(-1.0000, 0.0000)^T$

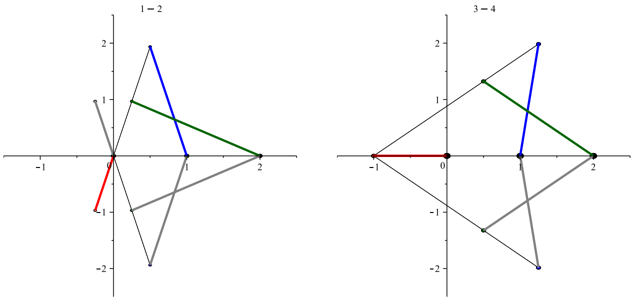


Fig. 7. The four moving platform poses obtained in Example IV and their graphical representation.

whose roots are $\frac{1}{4}$, 4, and $\frac{11}{2}$. However, note that the root at $T = \frac{1}{4}$ is outside the interval given by (16), therefore it does not correspond to a valid configuration. The moving platform poses for the case in which $\mathbf{p}_1 = (0, 0)^T$, $\mathbf{p}_2 = (1, 0)^T$, and $\mathbf{p}_3 = (2, 0)^T$ appear in Fig. 7.

If, in addition to the collinearity condition, the base and the moving platform are similar, the characteristic polynomial reduces to a polynomial of second degree.

E. Example V: Similar base and platform

In terms of the geometric parameters, the similarity constraint implies that $l_1 = kL_1$, $l_2 = kL_2$, and $l_3 = kL_3$, with $k > 0$. Substituting these expressions in $\Gamma(T)$, it reduces to a quartic. As an example of this analytic family, consider the robot presented in [8], whose geometric parameters are $\alpha > 0$, $\beta > 0$, $l_1 = 10\sqrt{5 - 2\sqrt{3}}$, $l_2 = 20$, $l_3 = 10$, $L_1 = 20\sqrt{5 - 2\sqrt{3}}$, $L_2 = 40$, $L_3 = 20$. Substituting these values in the resulting quartic, with input variables $\rho_1 = 2$, $\rho_2 = 44$, and $\rho_3 = 21$, the following characteristic polynomial is obtained:

$$1.7916T^4 - 2752.8830T^3 + 1.5749 \cdot 10^6 T^2 \\ - 3.9782 \cdot 10^8 T + 3.7457 \cdot 10^{10}.$$

The platform poses associated with each root of the above polynomial, for the case in which $\mathbf{p}_1 = (0, 0)^T$, $\mathbf{p}_2 = (40, 0)^T$, and $\mathbf{p}_3 = (10\sqrt{3}, 10)^T$, appear in Fig. 8.

F. Example VI: Mirrored base and platform

Consider the manipulator with geometric parameters $\alpha > 0$, $\beta < 0$, $l_1 = \sqrt{2}$, $l_2 = 1$, $l_3 = 1$, $L_1 = \sqrt{2}$, $L_2 = 1$, and $L_3 = 1$, and input joints $\rho_1 = 2$, $\rho_2 = \frac{1}{2}$, and $\rho_3 = 1$, the resulting irreducible characteristic polynomial in T is:

$$-32T^6 + 432T^5 - \frac{4455}{2}T^4 + \frac{39839}{8}T^3 \\ - \frac{339993}{64}T^2 + \frac{110565}{32}T - \frac{47385}{32}.$$

This example, which leads to a degeneration of Gosselin's formulation [7], corresponds to the robot presented in [19] where it is shown to be analytic. Indeed, it can be checked that the Galois group of the above polynomial is solvable.

T	θ	$\mathbf{p}_4$
329.4937	-1.5351	$(-0.7943, 1.8355)^T$
345.1531	1.5040	$(1.4501, 1.3774)^T$
387.5845	-1.5040	$(1.9805, -0.2787)^T$
474.3144	1.5351	$(1.0160, 1.7227)^T$

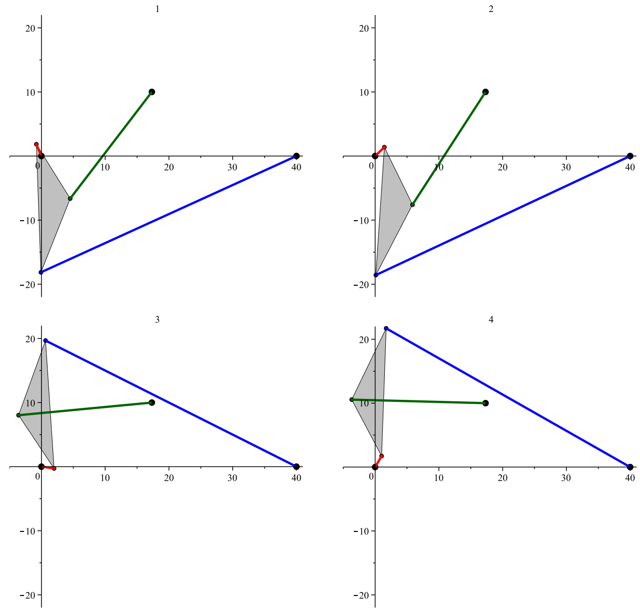


Fig. 8. The four moving platform poses obtained in Example V and their graphical representation.

VI. Conclusions

Stating the forward kinematics analysis of 3-RPR parallel planar robots directly in terms of poses introduces two major disadvantages: (a) reference frames have to be introduced, and (b) all formulas involve translations and rotations simultaneously. This paper proposes a different approach in which, instead of directly computing the sought Cartesian poses, a problem fully posed in terms of distances is first solved. Then, the original problem can be trivially solved by sequences of bilaterations.

All those formulations that include exceptions in their derivations lead to numerical instabilities when close to them. The formulation presented in this paper has no exceptions in its application.

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